Escape over a fluctuating barrier: Limits of small and large correlation times

Jan Iwaniszewski*
Institute of Physics, Nicholas Copernicus University, Grudziądzka 5, 87-100 Toruń, Poland
(Received 12 February 1996)

We investigate the problem of diffusion across a randomly fluctuating barrier in the presence of thermal noise. The barrier fluctuations are induced by an Ornstein-Uhlenbeck noise the strength Q of which is assumed to depend on the noise correlation time τ. In the vicinity of the limits of zero and infinite τ we calculate the exact formulas for the first two terms of the expansion in powers of τ of the mean first-passage time (MFPT) over the top of the barrier. The results are strongly conditioned by the form of the τ dependence of Q. The main conclusion is that the nonmonotonic τ dependence of the MFPT is generic, while the monotonicity of the MFPT occurs only in some specific cases. When τ increases from zero, for a class of barrier noises with Q increasing faster than linearly one should observe ''resonant activation,'' i.e., a minimum of the MFPT as a function of τ. The appearance of a maximum, called ''inhibition of activation,'' is also possible provided that the noise variance D increases faster than linearly as a function of $1/\tau$ in the vicinity of the limit $1/\tau \to 0$. Both kinds of extrema may also appear simultaneously. These effects depend neither on the shape of the barrier nor on its disturbance. If $Q(\tau)$ [or $D(1/\tau)$] varies linearly or slower as τ $(1/\tau)$ increases from zero, then the peculiarities of the perturbed barrier become essential and any type of τ dependence of the MFPT, also a monotonic one, is possible. The specific analogy between the properties of the MFPT for $\tau \to 0$ and for $\tau \to \infty$ is stressed. [S1063-651X(96)02909-1]

PACS number(s): 05.40.+j, 02.50.−r, 82.20.−w

I. INTRODUCTION

In recent years the stochastic dynamics community has become increasingly interested in noise-induced resonance-like effects in nonlinear systems. The best known and the most intensively studied phenomenon of this kind is *stochastic resonance* [1], a cooperative effect of nonlinearity, periodicity, and stochasticity, resulting in an enhancement of small coherent signals by noise. The conventional model of stochastic resonance [2] concerns diffusion over a barrier inside a symmetric double-well potential driven by a small asymmetric periodic signal which changes alternately the depths of the wells. Quite recently Doering and Gadoua [3] have discovered another resonancelike behavior for diffusion over a potential barrier with a randomly fluctuating height. The mean escape time $\mathcal{T}$ over the barrier has exhibited a minimum as a function of the correlation time τ of the barrier fluctuations. The minimal value of $\mathcal{T}$ has been of the order of τ which has suggested a resonancelike character of the phenomenon, hence the effect has been called *resonant activation* (RA).

Both phenomena are in fact some variants of one of the most fundamental problems in noisy dynamics, namely, the diffusive escape over a potential barrier. The foundations of its theory were laid by Kramers [4] half a century ago and since that time many modifications and generalizations of the problem have been formulated (see, e.g., [5]) leading to some interesting new phenomena such as the resonancelike behavior mentioned above. The ubiquity of noise-assisted barrier crossing in physics, chemistry, biology, and other branches of science or technology is such that one may expect many applications of any new effect associated with it.

The classical Kramers theory [4–6] deals with a diffusion induced by an idealized uncorrelated noise. As the main result one obtains the Arrhenius-like formula $\mathcal{T} \sim \exp(\Delta U/q)$ for the dependence of the escape time $\mathcal{T}$ on the barrier height ΔU and the noise strength q. A more realistic treatment of diffusion due to an exponentially correlated noise had not been investigated prior to the eighties. The general conclusion of any of the numerous theories [7,8] states that the noise memory slows the escape process down. In the case of a complex system whose dynamics is governed by a wide variety of time scales, it may happen that one or more of those time scales are comparable with the duration of the diffusion over the barrier. It is therefore reasonable to expect that during the barrier crossing event the barrier itself does not remain static — it will vary, being modulated by some relevant degree of freedom, often in a stochastic fashion. This may happen for some processes in complex systems like chemical reactions between large molecules [9] or for parametrically driven systems like dye lasers [7]. Some problems of the escape process in the presence of two noise sources (additive and multiplicative) have been considered in [10,11]. It seems that a systematic research of diffusion over a fluctuating potential barrier was initialized by Stein, Doering *et al.* [12,13] leading to the discovery of RA [3].

The toy model studied in [3] consists of a piecewise linear barrier the slope of which is randomly switching between two possible values. A simplified version of the problem has also been considered within the rate equation framework [14–18]. A more physical case deals with diffusion over a smooth potential barrier the shape of which varies randomly due to the fluctuations of a continuous parameter. Such a general model has been mentioned by Reimann [19] and discussed more carefully by Hänggi *et al.* [20–23] and Reimann [24,25]. Quite recently the first experimental results on an electronic analogue circuit have been announced by

*Electronic address: jiwanisz@phys.uni.torun.pl

Marchesoni *et al.* [26]. All those papers demonstrate the great complexity of the problem and formulate many open questions. It is the aim of this article to discuss some of them.

In the papers cited above the barrier fluctuations are supposed to be exponentially correlated. In calculations they are represented by two widely used kinds of noise, namely, a dichotomous noise (DN) or an Ornstein-Uhlenbeck noise (OUN). Both of them are parametrized by two quantities, [8] one of which is the correlation time τ, the control parameter of the problem. RA can appear in the presence of any of those noises, however, it strongly depends on the choice of the second, τ-independent parameter. E.g., if this parameter is chosen to be the noise variance then RA occurs; if it is the noise strength, RA is absent [19,26]. Moreover, Reimann [19] has shown that it is possible to obtain also the opposite effect, namely, a maximum of $\mathcal{T}(\tau)$ for a finite τ. The analysis of Hänggi [20] proves, that RA can occur generically whenever the colored noise intensity increases sufficiently fast with increasing τ, e.g., for a linear increase in the case when the noise variance is constant [see Eq. (2.3) below].

The discovery of RA caused some astonishment [27] since there existed a conviction about a monotonic τ dependence of the escape time, which had arisen after the investigation of diffusion in the presence of OUN [7,8]. We show, that the reason for this confusion is some arbitrariness in defining the OUN, namely, in the choice of the second relevant parameter of this noise.

Because of its non-Markovian character, the problem may be treated exactly only in some special cases, e.g., for a piecewise linear barrier disturbed by DN [3,14]. In general one needs some approximation [11,20,22,25]. However, it is not necessary to investigate the whole range of τ to anticipate the appearance of RA. It suffices to check the dependence of the escape time $\mathcal{T}$ on the correlation time for $\tau \to 0$. As follows from the study of the escape process induced by OUN, an increase in $\mathcal{T}$ with increasing τ seems to be a natural tendency. Hence the negative value of the first-order correction of $\mathcal{T}$ for infinitesimally small τ suggests the occurrence of RA [18,19,24,25]. Such an approach is applied in this paper.

We study the escape process calculating the mean first-passage time (MFPT) over the top of the barrier for a particle initially prepared in the bottom of the potential well. The non-Markovian character of the one-dimensional problem is avoided by embedding it in a two-dimensional Markovian process. In general, such a multidimensional problem is unsolvable analytically. However, since we are interested in the form of $\mathcal{T}(\tau)$ in the very vicinity of the white noise limit, the appropriate expansion results in some simple differential equations which yield the exact formulas by means of quadratures. Besides the white noise limit we study, also, in the same way the other limit $\tau \to \infty$, so we are able to predict any nonmonotonic behavior of MFPT associated either with a minimum (RA) or a maximum [we call this effect *inhibition of activation* (IA)] of the escape time $\mathcal{T}(\tau)$.

The outline of the paper is as follows. In Sec. II we analyze the relations between different quantities which characterize any OUN and we specify a class of noises which we use to disturb the barrier. Next (Sec. III) the dynamics of the problem is formulated and the general equations for the MFPT are given. In the two subsections we calculate the exact formulas for the MFPT, and its first-order correction, for both limits of the correlation time. A simplified version of those results is given in Sec. IV within the weak noise approximation, which allows one to analyze the effect of the shape of the barrier and of its perturbation on the escape process. A discussion of the τ dependence of the MFPT is presented in Sec. V. We find a relation between the properties of exponentially correlated fluctuations of the barrier and the possibility of the appearance of extrema of $\mathcal{T}(\tau)$. The main conclusion is that for almost any kind of exponentially correlated Gaussian noise one observes a generic nonmonotonicity of $\mathcal{T}$, either with a minimum, or with a maximum, or even with both extrema simultaneously. Some simple examples are given in Sec. VI while in Sec. VII we discuss the results and draw some conclusions.

II. EXPONENTIALLY CORRELATED NOISE

Let us consider a stationary Markovian Gaussian process $z(t)$ of vanishing mean and variance D. According to the Doob's theorem [28] this is necessarily an OUN which possesses an exponentially decreasing correlation function

$$C(t):=\langle z(t')z(t'+t)\rangle = D\,\exp\left(-\frac{|t|}{\tau}\right), \tag{2.1}$$

with the correlation time τ. This process is governed by the following linear stochastic differential equation:

$$\frac{dz}{dt} = -\frac{1}{\tau}z + \frac{\sqrt{2Q}}{\tau}\eta(t), \tag{2.2}$$

with $\eta(t)$ being a Gaussian white noise of zero mean and the correlation function $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. The noise strength Q is related to the variance D through the formula

$$D = Q/\tau. \tag{2.3}$$

The process $z(t)$ may also be treated as a superposition of harmonic oscillations with random amplitude and phase [28]. Its power spectrum

$$S(\omega):=\frac{1}{2\pi}\int_{-\infty}^{\infty} dt C(t)\exp(-i\omega t) = \frac{Q}{\pi}\frac{1}{1+\tau^2\omega^2} \tag{2.4}$$

is then the measure of the contribution of the oscillation of frequency ω to the total noise $z(t)$, while the inverse of the correlation time (the width of the spectrum) gives the range of the most essential oscillations.

The process $z(t)$ is completely characterized by two parameters. One of them is the correlation time τ, which is the crucial parameter in our considerations. The choice of the second parameter is somewhat arbitrary and depends on the details of the problem. Describing the properties of OUN we have used either the noise strength Q [(2.2) and (2.4)] or the noise variance D (2.1). Those two parameters seem to be the simplest, natural quantities which determine, together with τ, all the properties of $z(t)$.

Let us take a closer look at the behavior of the functions $C(t)$ and $S(\omega)$ when τ reaches its limiting values 0 or ∞. Keeping either Q or D constant one gets

$$
\begin{array}{lll}
 & Q = \text{const} & D = \text{const} \\
\tau \to 0 & C(t) \to Q\delta(t) & C(t) \to \begin{cases} D & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases} \\
 & S(\omega) \to Q/\pi & S(\omega) \to 0 \\
\tau \to \infty & C(t) \to 0 & C(t) \to D \\
 & S(\omega) \to \begin{cases} Q/\pi & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0 \end{cases} & S(\omega) \to D\delta(\omega).
\end{array} \tag{2.5}
$$

In the limit $\tau \to 0$, if Q is kept constant $z(t)$ becomes δ correlated and its variance D is infinite — $z(t)$ takes all values between minus and plus infinity with equal probability. Further, all the frequencies contribute with the same finite intensity Q/π, so the total power is infinite. These unphysical features reflect the extreme irregularity (e.g., nondifferentiability) of the Gaussian white noise [28]. If $D = \text{const}$ the properties of $z(t)$ are quite different. The possible values of $z(t)$ lie in a finite interval. Although the process is also memoryless and thus its spectrum is flat, the total power is finite, i.e., the intensity of any individual frequency vanishes. Consequently, the limit $\tau \to 0$ is in fact a noiseless one [28].

In the limit $\tau \to \infty$ the power spectrum consists of a single $S(\omega)$ component. For constant D the intensity of this component is infinitely large, but the total power is finite, so z is a time-independent number randomly distributed within a finite interval. If Q is constant then the amplitude $S(0)$ is finite and the total power vanishes. Consequently $D = 0$ and this is a noiseless case too [19] (freezing out of colored noise [23]).

The above analysis shows some analogy between the properties of $z(t)$ in both limits of τ. The correlation function $C(t)$ for $\tau \to 0$ (∞) behaves like the noise spectrum $S(\omega)$ for $\tau \to \infty$ (0). The same concerns the role of Q and D. Let us also notice, that depending on the choice of the constant parameter Q or D, in any limit one obtains either a singular noise with some nonphysical properties (infinite total power or infinite intensity of one spectral component) or a completely noiseless case.

In some large systems, when one represents the influence of the irrelevant degrees of freedom on the relevant part of the system by an OUN, the relation between the parameters of this noise may be much more complex, with neither Q nor D being independent of τ. In this paper we consider a general case of such a relation. Because we study the activation process in the presence of extremely fast or extremely slow barrier fluctuations, it suffices to specify the τ dependence of the second noise parameter in the limits of small and large τ only. In the following it is assumed that as $\tau \to 0$ or $\tau \to \infty$, the noise strength takes the form

$$
Q(\tau) = \tau^{\alpha}(Q_0 + \tau^{\beta} Q_1 + \cdots), \quad 0 < Q_0 < \infty \tag{2.6}
$$

where the parameters Q_i and the exponents α, β are generally different in both limits of τ. If only Q_0 is different from zero one recognizes the above mentioned constant-strength noise (CSN) and constant-variance noise (CVN) for $\alpha = 0$ and $\alpha = 1$, respectively. The form of expansion (2.6) yields $\beta \geqslant 0$ for $\tau \to 0$ or $\beta \leqslant 0$ for $\tau \to \infty$, while it follows from the analysis in Sec. III that $\alpha \geqslant 0$ or $\alpha \leqslant 1$, respectively. The latter inequalities have a simple explanation according to the discussion which follows Eq. (2.5). In the limit $\tau \to 0$, a negative value of α would mean that the variance D is infinitely large not only because the noise is δ correlated, but also due to the infinite value of its strength Q. This would imply that the intensity of any spectral component is infinite, hence, the total power would be ''doubly'' infinite. An analogous behavior when $\tau \to \infty$ is observed in the case $\alpha > 1$.

An analysis of the properties of $z(t)$ with $Q(\tau)$ given by (2.6) shows that $z(t)$ disappears unless $\alpha = 0$ or $\alpha = 1$ for $\tau \to 0$ or $\tau \to \infty$, respectively. Thus in both limits of τ the only nonvanishing members of the class of Gaussian stationary noises (2.2) are those which converge to CSN or CVN, respectively. Further, for a noise with $\alpha \neq 0$ $(\alpha \neq 1)$ any slight increase of τ $(1/\tau)$ from 0 means that the noise starts to have an effect on the system with which it is coupled. Because any characteristic time constant of the system is finite, in the neighborhood of $\tau = 0$ an increase of τ induces the same effect as an increase in the strength of the white noise. Similarly, for sufficiently large τ a decrease of τ implies the same changes as an increase of the variance of an infinitely long-correlated noise. Thus for the noises with $\alpha > 0$ $(\tau \to 0)$ or $\alpha < 1$ $(\tau \to \infty)$ it is enough to know the influence of zero- or infinitely long-correlated noises on the system to predict the role of the finite memory of $z(t)$.

In order to make Eq. (2.2), which describes the dynamics of barrier fluctuations, independent of the specific form of Q, we use the scaling

$$
z(t) = \sqrt{2Q/\tau}\, y(t), \tag{2.7}
$$

and consider in the following a Gaussian stationary noise $y(t)$ of zero mean and variance equal to 1/2, which is governed by the equation

$$
\frac{dy}{dt} = -\frac{1}{\tau} y + \frac{1}{\sqrt{\tau}} \eta(t). \tag{2.8}
$$

The noise strength Q appears explicitly in the fluctuating part of the potential in Eq. (3.1) which describes the escape process.

III. MEAN FIRST-PASSAGE TIME

Let us consider an overdamped motion of a particle in a potential which consists of two parts: a static one $U(x)$ and a time-dependent disturbance $z(t)V(x)$. The potential $U(x)$ has a local minimum at x_a and a local maximum at $x_b > x_a$, and for the sake of convenience there are no other extrema for $x < x_b$. Consequently there is a potential barrier of height $\Delta U := U(x_b) - U(x_a)$ with one metastable well on its left-hand side (lhs). Similarly we denote $\Delta V := V(x_b) - V(x_a)$ and we restrict ourselves to the case $\Delta V \geqslant 0$ [the negative sign could be absorbed into $z(t)$]. Further, it is assumed that the perturbation does not change the positions of the extrema of the total potential, so $V'(x_a) = V'(x_b) = 0$. To ensure the physical sense of the problem we also suppose that the disturbance is not very large so that the barrier is always present and the character of the extrema is preserved. The time-dependent factor $z(t)$ of the disturbance describes random fluctuations of the barrier and is realized by the OUN (2.2). The particle undergoes also thermal fluctuations characterized by a Gaussian white noise $\sqrt{2q}\xi(t)$ of vanishing mean, strength q, and correlation $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. Finally, we assume that $\xi(t)$ and $\eta(t)$ in (2.2) are uncorrelated. The dynamics of the particle is thus governed by the following Langevin equation:

$$\frac{dx}{dt} = -U'(x) - \sqrt{2\tau^{-1}Q(\tau)}V'(x)y(t) + \sqrt{2q}\xi(t), \tag{3.1}$$

where the scaled noise $y(t)$ (2.7) is used instead of $z(t)$.

The particle is initially located at the minimum x_a and the quantity of interest is its mean escape time $\mathcal{T}$ over the potential barrier. Among the standard approaches [5] to such a problem we choose the first-passage time (FPT) technique with $\mathcal{T}$ being the MFPT over the barrier top at x_b. For extremely short and for extremely long correlations of barrier fluctuations this approach results in some exact equations which are solvable by means of quadratures.

If the disturbance is absent, the MFPT for a static barrier reads [6]

$$T_s(x_a) = \frac{1}{q}\int_{x_a}^{x_b} du \frac{1}{\Psi_s(u)} \int_{-\infty}^{u} dv \Psi_s(v), \tag{3.2}$$

where

$$\Psi_s(x) = \exp\left(-\frac{U(x)}{q}\right). \tag{3.3}$$

The presence of correlated fluctuations of the barrier implies the main complication of the problem, namely, its being non-Markovian. We omit this difficulty by considering an equivalent two-dimensional Markovian process $[x(t),y(t)]$ governed by (3.1) and (2.8) for which the formalism of the Fokker-Planck equation is applicable. The Fokker-Planck operator associated with (3.1) and (2.8) reads

$$L(x,y) = \tau^{-1}L_0(y) + \sqrt{2\tau^{-1}Q(\tau)}yL_1(x) + L_2(x), \tag{3.4}$$

where

$$L_0(y) = \frac{\partial}{\partial y}y + \frac{1}{2}\frac{\partial^2}{\partial y^2}, \tag{3.5a}$$

$$L_1(x) = \frac{\partial}{\partial x}V'(x), \tag{3.5b}$$

$$L_2(x) = \frac{\partial}{\partial x}U'(x) + q\frac{\partial^2}{\partial x^2}. \tag{3.5c}$$

In the two-dimensional space the escape takes place when the particle crosses the separatrix which bounds the region of attraction of the potential minimum. Since $V'(x_b) = 0$ the separatrix is simply a straight line $x = x_b$. Hence we are sure that there is no ambiguity [29] in the definition of the escape moment — any event of passing the position x_b in the one-dimensional non-Markovian formulation is equivalent to crossing the separatrix $x = x_b$ in the two-dimensional Markovian one. The standard technique [6] yields an equation for the MFPT $T(x,y)$ over this separatrix

$$-1 = L^+(x,y)T(x,y), \tag{3.6}$$

with the absorbing barrier at $x = x_b$ imposing the condition

$$T(x_b, y) = 0. \tag{3.7}$$

The form of the operator L_0 suggests an expansion of $T(x,y)$ into a series of the Hermite polynomials $H_n(y)$ [30]

$$T(x,y) = \sum_{n=0}^{\infty} \tau^{p_n}T_n(x;\tau)H_n(y), \tag{3.8}$$

with the boundary conditions

$$T_n(x_b;\tau) = 0. \tag{3.9}$$

The exponents p_n, different in both limits of τ, give the leading dependence on the correlation time of the expansion coefficients in (3.8). After averaging (3.8) over the Gaussian distribution of y one obtains $\mathcal{T} = \tau^{p_0}T_0(x;\tau)$. Since the escape time should be well defined when the barrier fluctuations disappear (say $Q_0 = 0$), so $p_0 = 0$ for both $\tau \to 0$ and $\tau \to \infty$.

A. Small τ limit

Inserting (3.8) into (3.6) and using the properties of the Hermite polynomials one obtains an infinite set of equations for $T_n(x;\tau)$. The analysis of the dominant terms for small τ gives $\alpha \geqslant 0$ and $p_n = n(\alpha+1)/2$. Thus we get the set of equations (the dots stands for the higher-order terms in τ)

$$-1 = L_2^+T_0 + \tau^{\alpha}\sqrt{2Q_0}L_1^+T_1 + \tau^{\alpha+\beta}\frac{Q_1}{\sqrt{2Q_0}}L_1^+T_1 + \cdots, \tag{3.10a}$$

$$0 = -nT_n + \frac{1}{2}\sqrt{2Q_0}L_1^+T_{n-1} + \tau L_2^+T_n + \tau^{\alpha+1}(n+1) \times \sqrt{2Q_0}L_1^+T_{n+1} + \tau^{\beta}\frac{1}{2}\frac{Q_1}{\sqrt{2Q_0}}L_1^+T_{n-1} + \cdots,$$

$$\text{for } n = 1,2,\ldots, \tag{3.10b}$$

which may be solved perturbatively

$$T_n(x;\tau) = T_{n,0}(x) + \varepsilon T_{n,1}(x) + \cdots, \tag{3.11}$$

with the perturbation parameter ε being dependent on the values of α and β. A simple manipulation leads to the following equations for zeroth- and first-order terms of T_0:

for $\alpha = 0$:

$$\mathcal{L}_0^+T_{0,0} = -1, \tag{3.12a}$$

$$\mathcal{L}_0^+T_{0,1} = -Q_1L_1^{+2}T_{0,0}, \quad \varepsilon = \tau^{\beta}, \text{ for } 0<\beta<1 \text{ and } Q_1 \neq 0, \tag{3.12b}$$

$$\mathcal{L}_0^+ T_{0,1} = -Q_0 L_1^+ \mathcal{L}_0^+ L_1^+ T_{0,0} - Q_1 L_1^{+2} T_{0,0}, \quad \varepsilon = \tau, \quad \text{for } \beta = 1 \text{ and } Q_1 \neq 0, \tag{3.12c}$$

$$\mathcal{L}_0^+ T_{0,1} = -Q_0 L_1^+ \mathcal{L}_0^+ L_1^+ T_{0,0}, \quad \varepsilon = \tau, \quad \text{for } \beta > 1 \text{ or } Q_1 = 0; \tag{3.12d}$$

for $\alpha > 0$:

$$L_2^+ T_{0,0} = -1, \tag{3.12e}$$

$$L_2^+ T_{0,1} = -Q_0 L_1^{+2} T_{0,0}, \quad \varepsilon = \tau^\alpha. \tag{3.12f}$$

$\mathcal{L}_0$ is the Fokker-Planck operator of the total problem in the white noise limit $\tau \to 0$ (e.g., [7])

$$\mathcal{L}_0(x) = \frac{\partial}{\partial x}\left(U'(x) - \frac{1}{2}G'(x)\right) + \frac{\partial^2}{\partial x^2} G(x), \tag{3.13}$$

with the diffusion function

$$G(x) = q + Q(\tau = 0) V'^2(x). \tag{3.14}$$

$G(x)$ depends on α through the noise strength Q, namely, $Q(0) = Q_0$ for $\alpha = 0$, while $Q(0)$ vanishes for $\alpha > 0$. Thus the barrier fluctuations modify the diffusion function only for $\alpha = 0$ and this manifests itself in an increase of $G(x)$. If $\alpha > 0$ due to the disappearance of $z(t)$ (Sec. II) there is no effect of $V(x)$ on the escape event.

Equations (3.12) are solvable by means of quadratures with the boundary conditions similar to (3.9). The zeroth-order term, i.e., the exact result in the white noise limit, reads

$$T_{0,0} = \int_{x_a}^{x_b} du \frac{1}{\sqrt{G(u)}} \frac{1}{\Psi(u)} \int_{-\infty}^{u} dv \frac{1}{\sqrt{G(v)}} \Psi(v), \tag{3.15}$$

where

$$\Psi(x) = \exp\left(-\int^x dx' \frac{U'(x')}{G(x')}\right). \tag{3.16}$$

It follows from these formulas that the MFPT decreases when it is affected by barrier fluctuations ($\alpha = 0$). The first-order term reads

$$T_{0,1} = \mu(\alpha,\beta) A^0 + \nu(\alpha,\beta) B^0, \tag{3.17a}$$

where

$$A^0 = \int_{x_a}^{x_b} du\, U' \frac{V'^2}{G} + 2\int_{x_a}^{x_b} du \frac{1}{\sqrt{G}} \frac{1}{\Psi} \int_{-\infty}^{u} dv \frac{1}{\sqrt{G}} \Psi \int_v^u dw \left(\frac{U'V'}{G}\right)' \frac{qV''}{G} + \int_{x_a}^{x_b} du \frac{1}{\sqrt{G}} \frac{1}{\Psi} \int_{-\infty}^{u} dv \frac{1}{\sqrt{G}} \Psi \left[\frac{1}{2}\left(\frac{U'V' + qV''}{G}\right)^2\Big|_u + \frac{1}{2}\left(\frac{U'V' - qV''}{G}\right)^2\Big|_v + \left(\frac{qV''}{G}\right)^2\Big|_u + \left(\frac{qV''}{G}\right)^2\Big|_v\right], \tag{3.17b}$$

$$B^0 = -\int_{x_a}^{x_b} du \frac{1}{\sqrt{G}} \frac{1}{\Psi} \int_{-\infty}^{u} dv \frac{1}{\sqrt{G}} \Psi(v) \left[\frac{1}{2} \frac{V'^2}{G}\Big|_u + \frac{1}{2} \frac{V'^2}{G}\Big|_v + \int_v^u dw \frac{U'V'^2}{G^2}\right], \tag{3.17c}$$

and

$$\mu(\alpha,\beta) = \begin{cases} Q_0 & \text{for } \alpha = 0 \text{ and } (\beta \geq 1 \text{ or } Q_1 = 0), \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu(\alpha,\beta) = \begin{cases} Q_1 & \text{for } \alpha = 0 \text{ and } 0 < \beta \leq 1 \text{ and } Q_1 \neq 0, \\ Q_0 & \text{for } \alpha > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17d}$$

The index ''0'' of A^0 and B^0 indicates the zero-τ limit.

In B^0 one easily recognizes the first-order derivative of $T_{0,0}$ (3.15) with respect to the colored noise parameter Q_0. Since $U'(x) \geq 0$ for $x_a \leq x \leq x_b$ all the terms in the square brackets in (3.17c) are positive and B^0 is negative. As for A^0, the first and the third terms are positive, nevertheless the sign of the second one is not clear. Only the second term survives in the small noise limit (see Sec. IV), so a careful analysis of its sign is required. We return to this point further in Sec. IV.

B. Large τ limit

The case $\tau \to \infty$ may be treated in a similar way. The analysis of the dominant terms for large τ implies, that now $\alpha \leq 1$ and $p_n = n(\alpha - 1)/2$. Consequently one obtains

$$-1 = L_2^+ T_0 + \tau^{\alpha-1} \sqrt{2Q_0} L_1^+ T_1 + \tau^{\alpha+\beta-1} \frac{Q_1}{\sqrt{2Q_0}} L_1^+ T_1 + \cdots, \tag{3.18a}$$

$$0 = L_2^+ T_n + \frac{1}{2}\sqrt{2Q_0} L_1^+ T_{n-1} - \tau^{-1} n T_n + \tau^{\alpha-1} \times (n+1)\sqrt{2Q_0} L_1^+ T_{n+1} + \tau^\beta \frac{1}{2} \frac{Q_1}{\sqrt{2Q_0}} L_1^+ T_{n-1} + \cdots, \quad \text{for } n = 1,2,\ldots. \tag{3.18b}$$

It may be shown, however, that for $\alpha=1$ in order to calculate the zeroth-order term $T_{0,0}$ of the MFPT one needs to solve an infinite set of coupled equations for the zeroth-order terms of all the expansion coefficients $T_n(x;\tau)$ ($p_n=0$ for any n). This follows from the fact, that for very long-correlation times the noise variable $y(t)$ fluctuates adiabatically slowly in comparison to any other time scales of the problem. The influence of y on the dynamics of the system should be treated parametrically rather than perturbatively. Consequently one gets the zeroth-order approximation of the MFPT by averaging over the MFPT's calculated from (3.1) for fixed y. In the remaining case $\alpha<1$ the noise y enters the higher-order corrections only, so one may apply the expansion (3.8).

We begin with $\alpha=1$. The aim is to find the first two terms of the expansion

$$T(x,y)=T_0(x,y)+\varepsilon T_1(x,y)+\cdots \tag{3.19}$$

[compare (3.11)]. A simple calculation results in the following equations:

$$\mathcal{L}_\infty^+ T_0=-1, \tag{3.20a}$$

$$\mathcal{L}_\infty^+ T_1=-\frac{Q_1}{\sqrt{2Q_0}}L_1^+ yT_0,\quad \varepsilon=\tau^\beta,\quad \text{for}\ -1<\beta<0 \quad\text{and}\quad Q_1\neq 0, \tag{3.20b}$$

$$\mathcal{L}_\infty^+ T_1=-L_0^+T_0-\frac{Q_1}{\sqrt{2Q_0}}L_1^+ yT_0,\quad \varepsilon=\tau^{-1},\quad \text{for}\ \beta=-1 \quad\text{and}\quad Q_1\neq 0, \tag{3.20c}$$

$$\mathcal{L}_\infty^+ T_1=-L_0^+T_0,\quad \varepsilon=\tau^{-1},\ \text{for}\ \beta<-1\ \text{or}\ Q_1=0, \tag{3.20d}$$

with the zeroth-order (for the infinite correlation time) Fokker-Planck operator

$$\mathcal{L}_\infty=L_2+\sqrt{2Q_0}L_1 y. \tag{3.21}$$

The solution of (3.20a) reads

$$T_0(x_a,y)=\frac{1}{q}\int_{x_a}^{x_b}du\,\frac{1}{\Psi_s(u)}\int_{-\infty}^{u}dv\,\Psi_s(v)\times\exp\left(\frac{\sqrt{2Q_0}}{q}[V(u)-V(v)]y\right). \tag{3.22}$$

The parametrically treated noise y appears in an exponential form, so the averaging procedure over a Gaussian distribution is straightforward. One finds

$$T_0=\frac{1}{q}\int_{x_a}^{x_b}du\,\frac{1}{\Psi_s(u)}\int_{-\infty}^{u}dv\,\Psi_s(v)\times\exp\left(\frac{Q_0}{2q^2}[V(u)-V(v)]^2\right). \tag{3.23}$$

Similarly one calculates the averaged first-order term T_1:

$$T_1=\mu(\beta)A^\infty+\nu(\beta)B^\infty, \tag{3.24a}$$

where

$$A^\infty=-\frac{1}{q^4}\int_{x_a}^{x_b}du\,\frac{1}{\Psi_s(u)}\int_{-\infty}^{u}dv\,\Psi_s(v)\int_{v}^{x_b}du'\,\frac{1}{\Psi_s(u')}\times\int_{-\infty}^{u'}dv'\,\Psi_s(v')[V(u)-V(v)][V(u')-V(v')]\times\exp\left(\frac{Q_0}{2q^2}[V(u)-V(v)+V(u')-V(v')]^2\right), \tag{3.24b}$$

$$B^\infty(\alpha=1)=\frac{1}{2q^3}\int_{x_a}^{x_b}du\,\frac{1}{\Psi_s(u)}\int_{-\infty}^{u}dv\,\Psi_s(v)\times[V(u)-V(v)]^2\exp\left(\frac{Q_0}{2q^2}[V(u)-V(v)]^2\right), \tag{3.24c}$$

and

$$\mu(\beta)=\begin{cases}Q_0 & \text{for}\ \beta\leqslant -1\ \text{or}\ Q_1=0,\\ 0 & \text{otherwise}\end{cases}$$

$$\nu(\beta)=\begin{cases}Q_1 & \text{for}\ -1\leqslant\beta<0\ \text{and}\ Q_1\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.24d}$$

The index "∞" of A^∞ and B^∞ indicates the infinite-τ limit. As before, B^∞ is the first-order derivative of T_0 with respect to Q_0 and it is always positive. The sign of A^∞ is not clear and may depend on the shape of $V(x)$ (see Sec. IV).

If $\alpha<1$ the expansion into the Hermite polynomials (3.8) gives the following set of equations for the perturbative solution (3.11):

$$L_2^+T_{0,0}=-1, \tag{3.25a}$$

$$L_2^+T_{0,1}=-\sqrt{2Q_0}L_1^+T_{1,0}, \tag{3.25b}$$

$$L_2^+T_{1,0}=-\frac{1}{2}\sqrt{2Q_0}L_1^+T_{0,0},\quad \varepsilon=\tau^{\alpha-1}. \tag{3.25c}$$

In the limit $\tau\to\infty$ the escape time $T_{0,0}$ is not affected by the barrier fluctuations (see Sec. II) and it is given by (3.2), while its first-order correction $T_{0,1}$ for finite τ reads

$$T_{0,1}(x_a)=Q_0B^\infty(\alpha<1)=\frac{Q_0}{2q^3}\int_{x_a}^{x_b}du\,\frac{1}{\Psi_s(u)}\int_{-\infty}^{u}dv\,\Psi_s(v)[V(u)-V(v)]^2. \tag{3.26}$$

This expression is evidently positive.

IV. WEAK NOISE APPROXIMATION

In this section we approximate the general formulas derived in Sec. III by means of the saddle point method, valid for the weak noise limit when q and Q_0 are much smaller than the height of the potential barrier ΔU. This is a typical approximation exploited while investigating diffusion over a potential barrier. Nevertheless, we must stress that here the small noise limit is taken as the succeeding approximation. The preceding one is of course the limit $\tau\to 0$ or $\tau\to\infty$. The formulas for the MFPT and for its correction obtained in Sec. III are exact just for those limiting values of correlation time, while the approximation below is made for very small but finite q and Q_0. In this connection we consider the limits $\tau/q\to 0$ and $\tau/Q_0\to 0$ for $\tau\to 0$, or $q/\tau\to 0$ and $Q_0/\tau\to 0$ for $\tau\to\infty$. We do not investigate the other limits, like $q/\tau\to 0$ for $\tau\to 0$ (see, e.g., [31,7,8]), which possibly gives a different τ dependence of $\mathcal{T}$ [32].

For the white noise limit the saddle point method approximates the formulas (3.2), (3.15), (3.17b), (3.17c), respectively

$$T_s\approx\frac{\pi}{\sqrt{|U''(x_a)U''(x_b)|}}\exp\left(+\frac{\Delta U}{q}\right), \tag{4.1}$$

$$T_{0,0}\approx\frac{\pi}{\sqrt{|U''(x_a)U''(x_b)|}}\exp\left(+\int_{x_a}^{x_b}dx\,\frac{U'(x)}{G(x)}\right), \tag{4.2}$$

$$A^0\approx 2\left[\int_{x_a}^{x_b}du\left(\frac{U'V'}{G}\right)'\frac{qV''}{G}\right]T_{0,0}, \tag{4.3a}$$

$$B^0\approx-\left[\int_{x_a}^{x_b}du\,\frac{U'V'^2}{G^2}\right]T_{0,0}. \tag{4.3b}$$

Let us notice a great simplification of the expression for A^0, although its sign is still unknown — it depends on the form of $U(x)$ and $V(x)$.

To discuss this problem let us integrate by parts the right-hand side of (4.3a). This yields

$$A^0\approx 2q(I_1-I_2)T_{0,0}, \tag{4.4a}$$

where

$$I_1=2Q_0\int_{x_a}^{x_b}dw\,\frac{U'V'^2V''^2}{G^3}, \tag{4.4b}$$

and

$$I_2=\int_{x_a}^{x_b}dw\,\frac{U'V'V'''}{G^2}. \tag{4.4c}$$

Although the integral I_1 is always positive, nevertheless, it may be neglected as compared to I_2 when the strength of the correlated noise $z(t)$ is much smaller then the strength of $\xi(t)$, i.e., for $R:=Q_0/q\ll 1$.

The sign of the second integral I_2 depends on the forms of the potential $U(x)$ and its disturbance $V(x)$. In the whole interval of integration $U'(x)\geqslant 0$. In the vicinity of the points where $V'(x)$ reaches its extrema, the sign of $V'''(x)$ is opposite to the sign of $V'(x)$ and the integral (4.4c) over those regions (of type I) is negative. Between the extrema of $V'(x)$ there possibly exist some regions (of type II) where the signs of $V'(x)$ and $V'''(x)$ are the same and the integral (4.4c) is positive. The sign of I_2 depends on the relation between the contributions of those two regions. It seems that typically region I dominates since the absolute values of $V'(x)$ and $V'''(x)$ are greater in I than in II, and also since the region I always exists while the region II may be very small or may even be absent [e.g., for a cosinelike disturbance $V(x)\sim\cos(x)$ one obtains $V'(x)V'''(x)\sim-\sin^2(x)$]. Consequently $I_2<0$, so $A^0>0$.

There are two ways to increase the contribution of region II. The first one is to enlarge this region by an appropriate choice of $V(x)$. The second possibility is to choose the potential $U(x)$ in such a way, that it increases mainly within region II and $U'(x)$ reaches there its comparatively sharp maximum. The barrier in the potential of this kind is rather a steep one with a flat bottom or a flat top. On the other hand, the disturbance reaches its maximal value in region I, so it acts mainly on the flat parts of $U(x)$.

As an example let us consider a sixth-order-polynomial symmetric potential $U(x)=w[1/6x^6+1/4(p-1)x^4-1/2px^2]$. For $0\leqslant p\leqslant\infty$ and $w=12/(1+3p)$ it possesses two wells with the minima at $x=\pm 1$ and the height of the barrier $\Delta U=1$. This potential is much flatter than the mostly considered bistable quartic one [33]. The disturbance is given by a Lorentz-like function $V(x)=g/(g+x^2)$ which for small g is concentrated on the flat top of the barrier [34]. The long wings of this function guarantee the last required property, namely, the positiveness of the product $V'(x)V'''(x)$. For $p=0.02$ and $g=0.1$ one gets $I_2\approx+0.56$, and so it is possible to obtain a negative value of A^0, too.

One must notice, however, that the potential perturbation for the discussed case must be rather small because: (i) if R increases then I_1 becomes more important, (ii) the fluctuations of the potential cannot suppress the barrier between x_a and x_b. In our example we found that $|z(t)|$ should be less than 0.0113 (this is understood as the limitation for the central part of the Gaussian distribution, say a condition for its variance).

The weak noise approximation for an infinitely long correlation time gives for $\alpha<1$ the expression (4.1) for the zeroth-order term and

$$T_{0,1}=Q_0B^\infty(\alpha<1)\approx\frac{Q_0}{2q^2}(\Delta V)^2T_s \tag{4.5}$$

for its first-order correction (3.26) [35]. The case $\alpha=1$ requires a much more sophisticated analysis due to the appearance of the terms with $V(x)$ in the exponents of the exact formulas. In order to exploit the saddle point method in (3.23) and (3.24c) one has to find a maximum of the two-variable function

$$\Phi_2(u,v)=U(u)-U(v)+\tfrac{1}{2}R[V(u)-V(v)]^2. \tag{4.6}$$

It is located at the point $(u,v)=(x_b,x_a)$, as one expects, only if the matrix of second derivatives of the function $\Phi_2(u,v)$ is non-negatively defined at this point, i.e., if

$$U''(x_b)+R[V(x_b)-V(x_a)]V''(x_b)<0. \tag{4.7}$$

In this case one gets the formulas [35]

$$T_0 \approx \left(\left| 1 + R\Delta V \frac{V''(x_b)}{U''(x_b)} \right| \left| 1 + R\Delta V \frac{V''(x_a)}{U''(x_a)} \right| \right)^{-1/2} \times \exp\left(\frac{Q_0}{2q^2} (\Delta V)^2 \right) T_s , \tag{4.8}$$

$$B^{\infty}(\alpha = 1) \approx \frac{1}{2q^2} (\Delta V)^2 T_0 . \tag{4.9}$$

In [24,25] Reimann has specified three types of disturbances $V(x)$. Type I is a monotonic function with a maximum at $x = x_b$, so it heightens or lowers the barrier. Type II is defined by the equality $\Delta V = 0$ and it possesses a maximum inside the interval (x_a, x_b); this results in broadening or narrowing the barrier. The third mixed type of $V(x)$ possesses at least one extremum in (x_a, x_b) and different values at x_a and x_b, so $\Delta V \neq 0$. Such a disturbance changes simultaneously the height and the width of the barrier. The condition (4.7) is always fulfilled for the disturbance of type I or II, but it may be violated for the mixed III type. Namely, since $V'(x_b) = 0$ there is a minimum of $V(x)$ at $x = x_b$ and $V''(x_b) > 0$. If $V(x_b)$ is sufficiently large, the lhs of (4.7) may become positive. Consequently the maximum of $\Phi(u,v)$ is moved away from the point (x_b, x_a) to a new position at the point $(u,v) = (u_{max}, v_{max})$, where $x_a \leqslant v_{max} \leqslant u_{max} \leqslant x_b$.

If the position of the maximum of $\Phi_2(u,v)$ is known the approximation of the formulas (3.23) and (3.24c) is straightforward. We do not write here the corresponding expressions (compare [25]), but make only a remark. In the small noise limit the exponential term in (4.8) decides on the MFPT duration. Because the maximum of Φ_2 is moved away from the point (x_b, x_a), it follows that $\Phi_2(u_{max}, v_{max}) \geqslant \Phi_2(x_b, x_a) \geqslant \Delta U$. Consequently, the MFPT for type II potential is of the order of T_s, the MFPT for type I is greater and that for type III is greatest (assuming that ΔV is the same as for type I). This means that very long-correlated barrier fluctuations slow down the escape process if they disturb the height of the barrier. On the other hand they do not modify the escape time (in the zeroth-order term) if they alter the barrier width only. However, if the height and the width are disturbed simultaneously the escape time reaches its greatest value.

The application of the saddle point method to the formula (3.24b) is much more complicated. Because of the quadruple integral a maximum of the four variable function

$$\Phi_4(u,v) = U(u) - U(v) + U(u') - U(v') + \tfrac{1}{2} R [V(u) - V(v) + V(u') - V(v')]^2 \tag{4.10}$$

must be found. Similarly as before the expected position of the maximum is $(u,v,u',v') = (x_b, x_a, x_b, x_a)$, which results in

$$A^{\infty} \approx -\frac{1}{q^2} (\Delta V)^2 \exp\left[2 \frac{Q_0}{q^2} (\Delta V)^2 \right] T_0^2 , \tag{4.11}$$

so A^{∞} is negative [36].

In some cases the maximum may be moved from this point. A general analysis of such a possibility is very difficult, so we shall only discuss this topic qualitatively. The function Φ_4 is symmetric with respect to the replacement of the variables u and u'. This implies, that the maximum lies either at the point with $u = u_{max} = u' = u'_{max}$ or at the point $u = u_{max} \neq u' = u'_{max}$ as well as at the point $u = u'_{max} \neq u' = u_{max}$. The same property concerns the variables v and v'. Hence, there is either one maximum at the point $(u_{max}, v_{max}, u_{max}, v_{max})$, or a couple or even two couples of maxima placed symmetrically with respect to the line $u = u'$ or/and $v = v'$. In the case of a single maximum the saddle point method results in a negative A^{∞}, because the term $[V(u) - V(v)][V(u') - V(v')]$ in (3.24b) is positive. If there are two maxima at $u_{max} \neq u'_{max}$ and $v_{max} = v'_{max}$ (for simplicity we suppose that the maxima are far enough from each other) the approximation yields a sum of two identical terms which include products of the type $[V(u_{max}) - V(v_{max})][V(u'_{max}) - V(v_{max})]$. If, e.g., $V(u_{max}) < V(v_{max}) < V(u'_{max})$, such a product is negative and $A^{\infty} > 0$. Unfortunately we did not succeed in finding any example illustrating such a case. It seems, however, that this would take place only for very ''exotic'' $U(x)$ and $V(x)$, while for the ''ordinary'' potentials A^{∞} should be negative.

V. MFPT AS A FUNCTION OF THE CORRELATION TIME

In the previous sections we have derived the expressions for the MFPT and for its first-order correction in both limits of zero and infinite correlation time. Now, as mentioned in the Introduction, we are in a position to analyze the τ dependence of $\mathcal{T}$ and to discuss the effect of barrier fluctuations on the escape process, i.e., to find whether $\mathcal{T}(\tau)$ is monotonic or not. We must stress, however, that the form of $\mathcal{T}(\tau)$ may be very complicated, even with several extrema. This cannot be deduced only from the behavior of $\mathcal{T}(\tau)$ in the vicinity of the limiting values of τ, particularly because we do not specify the form of $Q(\tau)$ for all τ.

In order to simplify the notation of Sec. III, below we use $\mathcal{T}_0^i$ and $\mathcal{T}_1^i$ to denote the MFPT and its first-order correction, respectively. The index $i = 0, \infty$ designates one of the limits of τ.

A. Comparison of the MFPT for $\tau = 0$ and $\tau = \infty$

Let us first compare the values of MFPT in both limits of τ to find the ''natural'' expected tendency in the relation between $\mathcal{T}$ and τ. It follows from (3.15) that for $\tau = 0$ and $\alpha > 0$ the MFPT is equal to that of the stable barrier, which is a result of the disappearance of barrier fluctuations as $\tau \to 0$. If $\alpha = 0$ the potential disturbance does not disappear in the memoryless limit resulting in an increase of the diffusion function $G(x)$ and hence in a decrease of the MFPT. This property has been explained by Stein *et al.* [13] as follows. During the finite time interval needed for crossing the barrier there are some infinitesimally short $(\tau = 0)$ periods during which a random perturbation lowers the barrier below its unperturbed height. If until that moment the particle ''surmounts'' the barrier up to this level it gets free to evolve to the other side of the barrier. Hence, the average time of the process is smaller than that for the unperturbed barrier. The general relation for the whole class of noises (2.6) in the uncorrelated noise limit reads (compare [20])

$$\mathcal{T}_0^0 \leqslant T_s . \tag{5.1}$$

For the other limit $\tau \to \infty$, if $\alpha < 1$ the MFPT is equal to T_s (a noiseless case) while, as follows from (3.23), for $\alpha = 1$ the nondisappearing barrier fluctuations prolong the escape process. This fact is recognized in the literature too [12]. It ensues from the averaging over the barrier noise y (Sec. III.B.). The quantity $T_0(x,y)$ (3.22) to be averaged represents the MFPT over a perturbated barrier with a fixed value of y. Because y appears in (3.22) in an exponent the longer times associated with the higher barriers dominate in the averaged expression. The general inequality for the MFPT reads for this limit

$$T_s \leqslant \mathcal{T}_0^\infty . \tag{5.2}$$

From (5.1) and (5.2) one has (compare [25] for CVN)

$$\mathcal{T}_0^0 \leqslant \mathcal{T}_0^\infty . \tag{5.3}$$

In consequence the natural tendency is that the escape time is not decreased by the exponentially correlated barrier noise. If either $\alpha = 0$ for $\tau = 0$ or $\alpha = 1$ for $\tau = \infty$, one expects an increase of the MFPT with τ. If $\alpha \neq 0$ for $\tau = 0$ and $\alpha \neq 1$ for $\tau = \infty$, the MFPT is the same for both extreme values of τ and it is equal to that of a static barrier T_s. Nevertheless, since the dynamics does depend on the correlation time τ the MFPT cannot be a constant function of τ and so at least one extremum of $\mathcal{T}(\tau)$ occurs.

B. The dependence of the MFPT on small τ

The behavior of the MFPT in the very vicinity of the limiting values of τ is determined by the first-order term $\mathcal{T}_1^i$. Examine the limit $\tau \to 0$ first. The case $\alpha > 0$ is clear. It follows from (3.17) that $\mathcal{T}_1^0 < 0$. The correlations of barrier fluctuations reduce the time of diffusion across the barrier. This results from the nonexistence of barrier fluctuations for $\tau = 0$. Because for τ close to zero any time scale of the system is much greater than τ, the barrier noise $z(t)$ may be considered as an effectively white one [8]. The only consequence of an increase of τ is thus an increase in the strength $Q(\tau)$ yielding an increase in the diffusion function $G(x)$ and a decrease in $\mathcal{T}$. Since $\varepsilon = \tau^\alpha$ in (3.11) the smaller is the value of α the stronger is the reduction of the MFPT.

The case $\alpha = 0$ is much more intricate. If for small τ the strength $Q(\tau)$ increases faster than linearly, i.e., if $0 < \beta < 1$, [37] then $\varepsilon \mathcal{T}_1^0 = \tau^\beta Q_1 B^0$. Since $B^0 < 0$, for increasing τ the MFPT decreases for $Q_1 > 0$ and increases for $Q_1 < 0$. If $Q(\tau)$ varies more slowly than linearly $(\beta > 1)$ then $\varepsilon \mathcal{T}_1^0 = \tau Q_0 A^0$. The shapes of $U(x)$ and $V(x)$ become essential. We have analyzed this point in the weak noise approximation finding that usually $\mathcal{T}_1^0$ is positive and the MFPT increases with τ similarly as for diffusion driven by a colored noise. However, for some forms of $U(x)$ and $V(x)$, $\mathcal{T}_1^0$ may become negative. As mentioned in Sec. IV this is possible for a steep barrier with a flat bottom or a flat top, when a disturbance small enough acts on this flat parts of potential only. Due to the special relation between the derivatives of $U(x)$ and $V(x)$ it seems that the acceleration of the escape process comes out as an effect of a correlated perturbation in the region between the steep and flat parts of the barrier. This problem needs further study.

Finally, if $\beta = 1$ both terms (3.17b) and (3.17c) contribute to $\mathcal{T}_1^0$. It follows from the preceding discussion that they cooperate or compete in the determination of the sign of $\mathcal{T}_1$, so both an increase or a decrease of $\mathcal{T}$ are possible.

Irrespectively from the value of α one can distinguish two mechanisms of the influence of the correlated barrier fluctuations on the escape rate. The first one, described by B^0, connects the acceleration or slowing down of the escape process with an increase or a decrease of the strength $Q(\tau)$ of the disturbance noise, respectively. This behavior is independent of the shape of the barrier and its perturbation. However, if $Q(\tau)$ does not vary sufficiently rapidly when τ increases $(\alpha = 0$ and $\beta \geqslant 1)$ the details of the properties of $U(x)$ and $V(x)$ decide on the τ dependence of $\mathcal{T}$.

C. The dependence of the MFPT on large τ

The discussion of the other limit $\tau \to \infty$ proceeds quite similarly, however, as mentioned in Sec. II, it is more convenient to interpret the properties of $\mathcal{T}$ in terms of the variance D. The case $\alpha < 1$ is clear. It follows from (3.26) that $\mathcal{T}_1^\infty > 0$ — the longer the barrier fluctuations are correlated the weaker they impede the activation process. This is a consequence of the vanishing of $z(t)$ as $\tau \to \infty$. Even for very long but finite τ the barrier does fluctuate and the escape time over the heightened barrier contributes more substantially to the MFPT (see Sec. V.A.). As $\tau \to \infty$, the closer to 1 is the value of α, the slower is the decrease of D and the weaker is the decrease of $\mathcal{T}$.

In the case $\alpha = 1$ two terms A^∞ and B^∞ appear in the formula (3.24a) for $\mathcal{T}_1^\infty$. For $\beta \neq -1$ one of them dominates, while for $\beta = -1$ they both essentially contribute to $\mathcal{T}_1^\infty$. If $0 > \beta > -1$ then $\mathcal{T}_1^\infty = \tau^\beta Q_1 B^\infty$ and for increasing τ the MFPT decreases (increases) for $Q_1 > 0$ $(Q_1 < 0)$. For $\beta < -1$ the term A^∞ is the essential one and, as follows from Sec. IV, for not specially sophisticated cases it is negative, so $\mathcal{T}(\tau)$ increases while $\tau \to \infty$.

Quite similarly as in the case of $\tau \to 0$, irrespective of the value of α one can notice two ways of the influence of the stochastic disturbance on the considered phenomenon, however the analysis in terms of τ^{-1} is now more convenient. If $D(\tau^{-1})$ varies faster than linearly the behavior of $\mathcal{T}(\tau^{-1})$ reflects an increase or a decrease of the variance. For much smaller changes of $D(\tau^{-1})$ $(\alpha = 1$ and $\beta \leqslant -1)$ the details of the shapes of $U(x)$ and $V(x)$ decide about the properties of the MFPT.

It follows from the above discussion that a monotonic form of $\mathcal{T}(\tau)$ is possible only in a very specific case. Namely, for the zero-τ limit the barrier noise $z(t)$ cannot vanish $(\alpha = 0)$ and either $A^0 > 0$ for $\beta > 1$, or $Q_1 < 0$ for $0 < \beta < 1$, or $A^0 + Q_1 B^0 > 0$ for $\beta = 1$. In the infinite-τ limit the noise $z(t)$ must survive too $(\alpha = 1)$, and either $A^\infty < 0$ for $\beta < -1$, or $Q_1 < 0$ for $0 > \beta > -1$, or $A^\infty + Q_1 B^\infty < 0$ for $\beta = -1$. Let us notice, nevertheless, that these are only the necessary conditions. Some extrema of $\mathcal{T}(\tau)$, impossible to foresee by the present approach, might occur because of the specific properties of the system for finite τ. In the case of any other relation between the barrier noise parameters and the terms A^i and B^i, the MFPT either initially decreases in the vicinity of $\tau=0$, or finally decreases as $\tau\to\infty$. Due to (5.3) this implies a nonmonotonic behavior of $\mathcal{T}(\tau)$.

VI. EXAMPLES

In the preceding section we have discussed the general features of $\mathcal{T}$ for a class of noises with the strength given by (2.6). Let us illustrate the possible cases with some examples.

A. Noise of a constant strength CSN

This kind of noise is mostly used to mimic a stochastic signal with a finite memory and it is this noise which is usually called an Ornstein-Uhlenbeck one. For any τ its strength is constant $Q(\tau)=Q_0$, so $\alpha=0$ and $Q_1=0$. The limiting expressions for the MFPT are

$$\mathcal{T}=\mathcal{T}_0^0+\tau Q_0 A^0 \quad \text{for} \quad \tau\to 0,$$

$$\mathcal{T}=T_s+\tau^{-1}Q_0 B^\infty(\alpha<1) \quad \text{for} \quad \tau\to\infty, \tag{6.1}$$

with $\mathcal{T}_0^0$ given by (3.15). The quantity B^∞ is always positive, so as $\tau\to\infty$ the MFPT decreases towards its limiting value T_s. The sign of A^0 depends on $U(x)$ and $V(x)$. Usually it is positive and the MFPT increases with τ. In this case $\mathcal{T}(\tau)$ is surely a nonmonotonic function of τ with a maximum higher than the value of the escape time for the static barrier T_s. Thus IA occurs.

It follows from Sec. IV that A^0 may also be negative. In such a case $\mathcal{T}(\tau)$ initially decreases, reaches a minimum, then it increases to its maximum and finally it decreases towards T_s. The nonmonotonicity manifests itself in the existence of two extrema and both RA and IA occur.

The small-τ limit of the CSN has also been examined by Stein *et al.* [13]. They have found a linear increase of the mean exit time for increasing τ, however with some numerical integration as a final step in the theoretical analysis. Such a methodology could be the reason for the absence of the possibility of $\mathcal{T}$ decreasing which, as we have shown, occurs only for some special types of potentials.

B. Noise of a constant variance CVN

A CVN is defined by $Q(\tau)=\tau Q_0$ for any τ, so $\alpha=1$ and $Q_1=0$. The expressions for the MFPT are as follows:

$$\mathcal{T}=T_s+\tau Q_0 B^0 \quad \text{for} \quad \tau\to 0,$$

$$\mathcal{T}=\mathcal{T}_0^\infty+\tau^{-1}Q_0 A^\infty \quad \text{for} \quad \tau\to\infty, \tag{6.2}$$

with $\mathcal{T}_0^\infty$ given by (3.23). Since B^0 is negative, the MFPT initially decreases. It follows from Sec. IV that A^∞ is usually negative and $\mathcal{T}(\tau)$ increases up to $\mathcal{T}_0^\infty$ when $\tau\to\infty$. This means that one minimum of $\mathcal{T}(\tau)$ does exist and RA takes place [21,22,24–26]. Let us notice, that a dichotomous noise also possesses a constant variance and the appearance of RA in the presence of DN disturbance [3,14,17,21] is absolutely consistent with the present results.

Considering the long τ limit for $\alpha=1$ in Sec. IV we have not been able to exclude an existence of some ''exotic'' potentials for which $A^\infty>0$, so $\mathcal{T}(\tau)$ should exhibit both a minimum and a maximum. Such a case cannot appear in Reimann's approach [25], in which under very general conditions for $U(x)$ and $V(x)$ the MFPT monotonically increases with τ. However, the so called ''kinetic equation'' [Eq. (4.2) in [25]] being the basis of the consideration does not seem to be systematic, because the loss rate $k(y)$ used there has been taken just in the limit $\tau\to\infty$ without any corrections for finite τ. If one regards this correction it will be an open question whether the MFPT remains increasing in any case.

C. Noise vanishing for both limits of τ

Now we consider an intermediate case, namely, a noise also with $Q_1=0$, but with $0<\alpha<1$, say $\alpha=0.5$. Since both the noise strength for $\tau\to 0$ as well as the variance for $\tau\to\infty$ vanish, barrier fluctuations modify the dynamics only when the correlation time is finite. The formulas for the MFPT for both limits of τ read

$$\mathcal{T}=T_s+\tau^{1/2}Q_0 B^0 \quad \text{for} \quad \tau\to 0,$$

$$\mathcal{T}=T_s+\tau^{-1/2}Q_0 B^\infty(\alpha<1) \quad \text{for} \quad \tau\to\infty. \tag{6.3}$$

Since $B^0<0$ and $B^\infty>0$, in both limits of small and large correlation time $\mathcal{T}(\tau)$ decreases with increasing τ. Both limiting values of the MFPT are the same so there are two extrema in the form of $\mathcal{T}(\tau)$ — a minimum for the smaller and a maximum for the greater value of τ. Both phenomena RA and IA appear and this feature is completely independent of the shapes of the barrier and its disturbance.

D. Nonvanishing noise in any of the limits of τ

As the fourth example we take a noise which does not vanish in any of the limits of τ. We choose a noise strength of the form $Q(\tau)=Q_0(1+\tau)$. For $\tau\to 0$ the noise parameters are $Q_1=Q_0>0$, $\alpha=0$, $\beta=1$, while for $\tau\to\infty$ they are: $Q_1=Q_0>0$, $\alpha=1$, $\beta=-1$. The expressions for the MFPT are as follows:

$$\mathcal{T}=\mathcal{T}_0^0+\tau Q_0(A^0+B^0) \quad \text{for} \quad \tau\to 0,$$

$$\mathcal{T}=\mathcal{T}_0^\infty+\tau^{-1}Q_0[A^\infty+B^\infty(\alpha=1)] \quad \text{for} \quad \tau\to\infty, \tag{6.4}$$

with $\mathcal{T}_0^0$ given by (3.15) and $\mathcal{T}_0^\infty$ by (3.23). Depending on the case, one expects a variety of behaviors of $\mathcal{T}(\tau)$. For the special kinds of potentials mentioned in Sec. IV, when $A^0<0$ and $A^\infty>0$ both kinds of extrema appear. However, this is also possible for ''normal'' potentials for which $A^0>0$ and $A^\infty<0$, namely, when $A^0<-Q_0B^0$ and $A^\infty>-Q_0B^\infty$. If one of these inequalities is false, either a maximum or a minimum appears. Only if $A^0>-Q_0B^0$ and $A^\infty<-Q_0B^\infty$, one may expect a monotonic increase of $\mathcal{T}(\tau)$, though this is not yet a sufficient condition.

VII. DISCUSSION

The non-Markovian character of the diffusional escape over a fluctuating barrier, when those fluctuations are correlated, implies that an exact analysis of the global problem is impossible. One may investigate rigorously two limiting cases, namely, those with infinitely short and infinitely long correlated fluctuations. It follows from the previous sections, that this suffices to find many interesting aspects of the subject. The main result of our considerations is that the nonmonotonic τ dependence of $\mathcal{T}$ is generic, while the expected strictly monotonic increase appears for very specific kinds of Ornstein-Uhlenbeck-like noises (e.g., example D in Sec. VI). This is a consequence of the disappearance of the barrier noise $z(t)$ in one or in both limits of τ for most of the members of the class (2.6) of Gaussian noises (2.2), among others for the mostly used CSN and CVN. The existence of either a minimum or a maximum (or both of them) of $\mathcal{T}(\tau)$ indicates the occurrence of the phenomenon of resonant activation or inhibition of activation [38] (or both of them), respectively.

For most of the cases of the Gaussian noise $z(t)$ the behavior of $\mathcal{T}(\tau)$ for small τ is determined only by the properties of $z(t)$ and depends neither on the shape of the barrier nor on its disturbance. Nevertheless, there are two cases in which those shapes play an important role. First, it is the case discussed in Sec. IV when not too strong fluctuations of the flat part of the barrier may accelerate the escape event. The second case appears for $\alpha=0$ and $\beta=1$ when the two contributions A^0 and B^0 to the first-order correction of MFPT (3.17a) are of opposite signs. Quite similarly, for $\tau\to\infty$, the dependence of $\mathcal{T}(\tau)$ on τ is usually related only to the properties of $z(t)$. The forms of $U(x)$ and $V(x)$ become important merely for $\alpha=1$ and $\beta=1$, i.e., when both terms A^∞ and B^∞ contribute to $\mathcal{T}_1^\infty$

In Sec. II we have noticed an analogy between the properties of the Gaussian correlated noise in both limits of τ. This may be extended to the properties of the MFPT for the escape over a fluctuating barrier. The possibility of the occurrence of a minimum of $\mathcal{T}(\tau)$ depends on the behavior of the noise strength $Q(\tau)$ in the zero-τ limit while the appearance of a maximum results from the properties of the noise variance $D(\tau)$ for an infinitely large τ. It seems that this symmetry is disturbed only for the special forms of potentials discussed in Sec. IV. This case has no counterpart in the infinite-τ limit, however the sign of (3.24b) remains an open question and one cannot exclude some analogy in this case, too.

In [20] Hänggi has found that RA appears whenever $Q(\tau)$ increases sufficiently rapidly with increasing τ. Our results state precisely the character of this τ dependence of the noise strength. Namely, RA occurs if: (i) $Q(0)=0$ which means that for any $\tau>0$ the strength $Q(\tau)$ is greater than $Q(0)$, (ii) $Q(0)\neq 0$ and $Q(\tau)$ increases faster than linearly [37] ($\alpha=0$ and $0<\beta<1$ with $Q_1>0$). If the increase of $Q(\tau)$ is slower then RA usually does not occur unless $U(x)$ and $V(x)$ have some special properties.

In the above conclusions we do not mention the existence of the thermal noise $\xi(t)$. In fact the character of the τ dependence of the MFPT is influenced by the white noise strength q in two special cases, only. The first one is the case of competition between the terms A^i and B^i when their exact values are important. The second one is the case of the special shapes of $U(x)$ and $V(x)$ discussed in Sec. IV, for which the sign of A^0 depends on the ratio R of the two noise strengths. If q is sufficiently small there is no possibility for the RA to appear. Besides those exceptions, since $V'(x)$ vanishes at x_a and x_b, the white noise $\xi(t)$ is essential only for the initialization of the evolution from the bottom of the well as well as for a successful surmount at the very top of the barrier. The value of q governs the time scale of the process rather, than decides about the way in which barrier fluctuations change the rate of the escape.

In the present paper the escape process has been characterized by the mean time $\mathcal{T}$ of the first arrival at the barrier top. In order to get the escape time one must be sure that the escape event really takes place and the particle will not return to the initial well immediately. Thus one has to multiply $\mathcal{T}$ by the inverse of the probability of leaving the barrier top in the required direction outside the potential well. For a symmetric barrier this probability equals 1/2. In an unsymmetrical case it is conditioned by the details of the problem and possibly depends also on τ. It seems reasonable, however, that the τ dependence of the activation process introduced in such a way plays a secondary role, if any, because it is related to the relaxation from an unstable state rather than to the escape over the barrier. The most crucial τ dependence of the escape time, which is observed for both symmetric and unsymmetrical barriers, comes from the τ dependence of the MFPT over the top of the barrier, being considered in this paper.

Further, we have used the assumption that the positions of the extrema of the total fluctuating potential are fixed. This has allowed for a direct application of the FPT technique with both the initial point x_a and the threshold x_b being well defined. The omission of this restriction $[V'(x_a)\neq 0$ or $V'(x_b)\neq 0]$ would involve a necessity of averaging over an ensemble of initial conditions as well as a more complicated form of the separatrix. Nevertheless, the exact formulas of Sec. III suggest that even then most of our conclusions would remain valid. One should possibly expect more difficulties while analyzing the signs of A^0 and A^∞. Also, if the integration interval in (3.17c) or (4.3b) is extended outside the interval (x_a,x_b) then the presence of $U'(u)$ in those formulas yields some trouble in determining the sign of B^0. Besides one must notice that this generalization concerns the variation of the distance between the extrema of the global potential rather than the modifications of the height of the barrier. This implies that the effect of the potential fluctuations would appear in the prefactor of the formula for the escape time and not in its exponent.

Our calculations for the limiting values of τ do not allow us to conclude anything about the place and size of the extrema of $\mathcal{T}$. To this end one needs an approximation which deals with finite values of τ. Such approach has been proposed by Madureira *et al.* [22] with a good agreement between theory and numerics. We must notice, however, that their generalized unified colored noise theory seems not to be correctly formulated in the limit of large τ. Namely, it states that if barrier fluctuations are generated by CSN then the escape time increases up to T_s as $\tau\to\infty$, while we prove here that T_s should be approached from above. This inconsistency appears because in [22] some terms of the order of τ^{-1} related to the barrier noise have been omitted in the large-τ Markovian approximation of the problem (see the discussion at the beginning of Sec. V A of [22]).

Finally let us notice that the existing numerical [25,23] and experimental [26] data do not confirm so far the appearance of the maximum of $\mathcal{T}$. The possible explanation of this fact seems to be the choice of the perturbating potential in the cited references, namely $V'(x)=x$. Such a disturbance dramatically changes the position of the potential minimum and has been excluded in the present model. This point needs further studies.

ACKNOWLEDGMENT

The author would like to express his gratitude to Professor S. T. Dembiński for critically reading the manuscript.

[1] *Proceedings of the NATO Advanced Research Workshop on Stochostic Resonance in Physics and Biology*, edited by F. Moss, A. Bulsara, and M. F. Shlesinger [J. Stat. Phys. **70**, Nos. 1-2 (1993)]; P. Jung, Phys. Rep. **234**, 175 (1993); F. Moss, in *Contemporary Problems in Statistical Physics*, edited by G. H. Weiss (SIAM, Philadelphia, 1994), p. 205; *Proceedings of the International Workshop on Fluctuations in Physics and Biology: Stochostic Resonance, Signal Processing and Related Phenomena*, edited by A. Bulsara *et al.* [Nuovo Cimento **17D**, Nos. 7-8 (1995)].

[2] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[3] Ch. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992).

[4] H. A. Kramers, Physica **7**, 284 (1940).

[5] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[6] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[7] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH, New York, 1990).

[8] P. Hänggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995).

[9] J. Wang and P. Wolynes, Chem. Phys. **180**, 141 (1994).

[10] P. Hänggi, Phys. Lett. A **78**, 304 (1980).

[11] K. M. Rattray and A. J. McKane, J. Phys. A **24**, 1215 (1991).

[12] D. L. Stein, R. G. Palmer, J. L. van Hemmen, and C. R. Doering, Phys. Lett. **136**, 353 (1989).

[13] D. L. Stein, Ch. R. Doering, R. G. Palmer, J. L. Van Hemmen, and R. M. McLaughlin, J. Phys. A **23**, L203 (1990).

[14] U. Zürcher and Ch. R. Doering, Phys. Rev. E **47**, 3862 (1993).

[15] C. Van den Broeck, Phys. Rev. E **47**, 4579 (1993).

[16] A. Fuliński, Phys. Lett. A **180**, 94 (1993).

[17] M. Bier and R. D. Astumian, Phys. Rev. Lett. **71**, 1649 (1993).

[18] J. J. Brey and J. Casado-Pascual, Phys. Rev. E **50**, 116 (1994).

[19] P. Reimann, Phys. Rev. E **49**, 4938 (1994).

[20] P. Hänggi, Chem. Phys. **180**, 157 (1994).

[21] P. Pechukas and P. Hänggi, Phys. Rev. Lett. **73**, 2772 (1994).

[22] A. J. R. Madureira, P. Hänggi, V. Buonomano, and W. A. Rodrigues Jr., Phys. Rev. E **51**, 3849 (1995); **52**, 3301(E) (1995).

[23] R. Bartussek, P. Hänggi, and A. J. R. Madureira, Phys. Rev. E **52**, 2149 (1995).

[24] P. Reimann, Phys. Rev. Lett. **74**, 4576 (1995).

[25] P. Reimann, Phys. Rev. E **52**, 1579 (1995).

[26] F. Marchesoni, L. Gammaitoni, E. Menichella-Saetta, and S. Santucci, Phys. Lett. A **201**, 275 (1995).

[27] J. Maddox, Nature **359**, 771 (1992).

[28] W. Horsthemke and R. Lefever, *Noise–Induced Transitions* (Springer, Berlin, 1984).

[29] P. Hänggi, P. Jung, and P. Talkner, Phys. Rev. Lett. **60**, 2804 (1988); C. R. Doering, P. S. Hagan, and C. D. Levermore, *ibid.* **60**, 2805 (1988).

[30] The expansion used in this paper is somewhat related to the singular perturbation technique exploited in [13] [see also [28] or C. R. Doering, P. S. Hagan, and C. D. Levermore, Phys. Rev. Lett. **59**, 2129 (1987)]. Nonetheless since we have to solve an equation for the mean FPT and not for the probability distribution of FPT we avoid the fundamental difficulty of that method, namely, the noninvertibility of the operator L_0.

[31] M. M. Klosek-Dygas, A. J. Matkowsky, and Z. Schuss, Phys. Rev. A **38**, 2605 (1988).

[32] The case $q/\tau\to 0$ for $\tau\to 0$ has been considered for CSN in [11] by means of the path integral theory. The conclusion states that for small τ the mean escape time increases as τ^2.

[33] J. Iwaniszewski, P. V. E. McClintock, and N. D. Stein, Phys. Rev. E **50**, 3538 (1994).

[34] The Lorentz-like potential $V(x)$ does not fulfill the requirement that $V'(x_a)=0$, however for the given values of parameters and for small Q_0 this is negligible.

[35] If $\Delta V=0$ the higher-order term of the saddle point method must be considered. Due to the form of the exact expressions (3.24c) and (3.26), however, the sign of B^∞ remains always positive.

[36] Because for $u=u'$ and $v=v'$ the integrated function in (3.24b) is always positive this conclusion is valid for all types of $V(x)$, even though for $\Delta V=0$ the higher-order term of the saddle point method has to be calculated.

[37] Note that for $\tau<\beta^{1/(1-\beta)}$ the increase of τ^β is faster than linear.

[38] It is a matter of further studies whether the appearance of a maximum of $\mathcal{T}(\tau)$ [called here an inhibition of activation (IA)] is also, by analogy to RA, a resonancelike phenomenon, so it should rather be called a *resonant inhibition of activation.*